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OSCILLATIONS IN AN $x^{(2m+2)/(2n+1)}$ POTENTIAL

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Recently, Mickens [1] studied a class of non-linear, one-dimensional oscillators in an $x^{(2n+2)/(2n+1)}$ potential. He obtained approximate solutions by using harmonic balance method. Motivated by Mickens' paper, we will extend his results in this study.

Consider a one-dimensional oscillation whose potential takes the form

$$V(x) = V_0 x^{(2m+2)/(2n+1)}, \tag{1}$$

where $m \geqslant n$ and $m, n = 0, 1, 2, \ldots$, and V_0 is a positive constant. The force derived from equation (1) is

$$f(x) = -\frac{\mathrm{d}V}{\mathrm{d}x} = -\left(\frac{2m+2}{2n+1}\right) V_0 x^{(2(m-n)+1)/(2n+1)}. \tag{2}$$

A particle of mass, M, acted on by the force of equation (2), has the equation of motion

$$M\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \left(\frac{2m+2}{2n+1}\right) V_0 x^{(2(m-n)+1)/(2n+1)} = 0. \tag{3}$$

By a proper change of both the dependent and independent variables, this equation can be transformed to dimensionless form

$$\frac{\mathrm{d}^2\bar{x}}{\mathrm{d}\bar{t}^2} + \bar{x}^{(2(m-n)+1)/(2n+1)} = 0. \tag{4}$$

In the work to follow, the "bars" will be dropped to give

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + x^{(2(m-n)+1)/(2n+1)} = 0. \tag{5}$$

For $m = n = 0$, this equation becomes

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + x = 0, \tag{6}$$

which is a linear oscillator. For $m = 1$ and $n = 0$, equation (5) gives

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + x^3 = 0. \tag{7}$$

If $m = n$, equation (5) becomes

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + x^{1/(2n+1)} = 0, \tag{8}$$

which has been studied in reference [1]. When $m = 3$ and $n = 1$, equation (5) yields

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + x^{5/3} = 0. \tag{9}$$

The following arguments are similar to Mickens' [1].

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The system equations for equation (5) are

$$\frac{\mathrm{d}x}{\mathrm{d}t} = y, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = -x^{(2(m-n)+1)/(2n+1)} \tag{10}$$

and the first order differential equation that the trajectories in the (x, y) phase-space satisfy is

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x^{(2(m-n)+1)/(2n+1)}}{y}. \tag{11}$$

Since $x^{(2(m-n)+1)/(2n+1)}$ is an odd function of x, equation (11) is invariant under the three transformations:

$$T_1 : x \to -x, \quad y \to y, \qquad T_2 : x \to x, \quad y \to -y, \tag{12a, b}$$

$$T_3 : x \to -x, \quad y \to -y. \tag{12c}$$

The corresponding null-clines [2], curves along which the slope $\mathrm{d}y/\mathrm{d}x$ is either zero or unbounded, are

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 0 : \quad x = 0 \quad \text{or along the } y\text{-axis}, \tag{13a}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \infty : \quad y = 0 \quad \text{or along the } x\text{-axis}. \tag{13b}$$

The results given in equations (12) and (13) are exactly the same as those for the simple harmonic oscillator [2, 3]. Consequently, applying the standard phase-space qualitative methods [2–4], it can be concluded that all the trajectories in phase-space are closed [2]. This implies that all the solutions to equation (5) are periodic [2–4].

In order to obtain an approximate solution, we use the method of harmonic balance [2]. Equation (5) can be rewritten as

$$\left(\frac{\mathrm{d}^2x}{\mathrm{d}t^2}\right)^{2n+1} + x^{2(m-n)+1} = 0, \quad x(0) = x_0, \quad \dot{x}(0) = 0, \tag{14}$$

where x_0 is given and the approximate solution is taken to be

$$x(t) \approx A \cos \omega t. \tag{15}$$

The parameters A and ω can be determined from the harmonic balance procedure [2]. Substitution of equation (15) into equation (14) gives

$$(-A\omega^2 \cos \theta)^{2n+1} + (A \cos \theta)^{2(m-n)+1} \approx 0, \quad \theta = \omega t. \tag{16}$$

Using the relation [5]

$$(\cos \theta)^{2n+1} = \left(\frac{1}{2^{2n}}\right) \sum_{k=0}^{n} \binom{2n+1}{k} \cos[2(n-k)+1]\theta \tag{17}$$

and keeping only the term in $\cos \theta$ allows the following result to be obtained:

$$A\left[\left(\frac{A}{2}\right)^{2(m-n)} \binom{2(m-n)+1}{m-n} - \left(\frac{A}{2}\right)^{2n} \binom{2n+1}{n} \omega^{4n+2}\right] \cos \theta + (\text{higher order harmonics}) = 0. \tag{18}$$

This leads to the following approximate solution for equation (5):

$$x(t) \approx x_0 \cos[\omega_n(x_0)t], \tag{19}$$

where

$$\omega_n(x_0) = \left[\frac{2^{2(2n-m)} \begin{pmatrix} 2(m-n)+1 \\ m-n \end{pmatrix}}{x_0^{2(2n-m)} \begin{pmatrix} 2n+1 \\ n \end{pmatrix}} \right]^{1/(4n+2)}. \tag{20}$$

If $m = n$, equation (20) becomes

$$\omega_n(x_0) = \left[\frac{2^{2n}}{x_0^{2n} \begin{pmatrix} 2n+1 \\ n \end{pmatrix}} \right]^{1/(4n+2)}, \tag{21}$$

which is equation (24) in reference [1]. For equation (7) ($m = 1$ and $n = 0$), equation (20) gives

$$\omega_n(x_0) = \frac{\sqrt{3}}{2} x_0, \tag{22}$$

which is a well-know result [2]. For equation (9) ($m = 3$ and $n = 1$), expression (20) becomes

$$\omega_n(x_0) = (\tfrac{5}{6} x_0^2)^{1/6}. \tag{23}$$

For linear equation (6) ($m = n = 0$), from equation (20) we have

$$\omega_n(x_0) = 1. \tag{24}$$

In summary, the mathematical properties of the oscillator, given by equation (5), have been studied. All the solutions are found to be periodic and the method of harmonic balance was used to construct an analytical approximation to these solutions. Therefore, the results in reference [1] have been extended.

Finally, in order to analyse the accuracy of the approximate solution (19), we take equation (9) for example. Equation (9), which has an approximate solution

$$x(t) = x_0 \cos\left[\left(\frac{5x_0^2}{6} \right)^{1/6} t \right], \tag{25}$$

was studied numerically using the non-standard finite difference schemes of Mickens [6]. The particular discrete model used was

$$\frac{x_{k+1} - x_k}{\sin(h)} = y_k, \qquad \frac{y_{k+1} - y_k}{\sin(h)} = -(x_{k+1})^{5/3}, \tag{26}$$

where $\phi(h) = \sin(h)$ is the so-called denominator function, with h being the time step-size. All numerical solutions were found to be periodic with closed trajectories in phase-space. Figure 1 gives plots of the numerical solution x_k versus t (solid curve) and the approximate solution (25) versus t (dotted curve) for $h = 0{\cdot}02$, $k = 0, 1, \ldots, 500$ (i.e., $0 \leqslant t \leqslant 10$), $\dot{x}(0) = 0$ and $x(0) = 10$, 100, 500 respectively. From Figure 1, we see that the numerical solutions and approximate solutions are close to each other, especially when t is small.

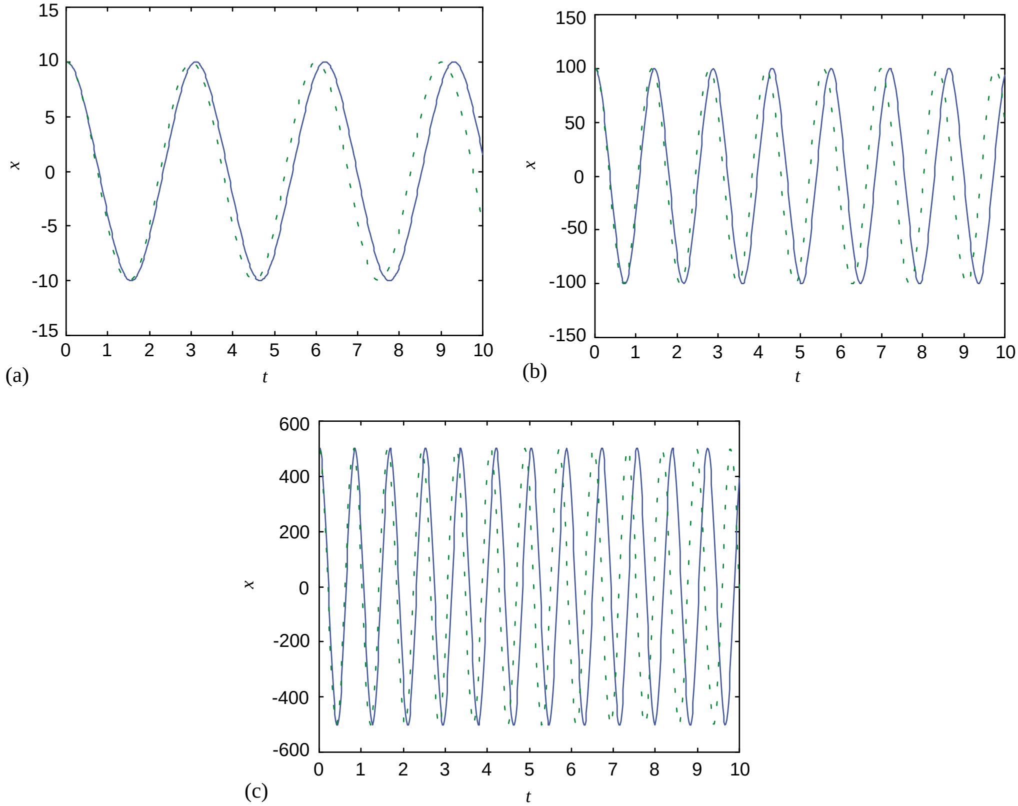

Figure 1. Plots of the numerical solution x_k versus t (solid curve) and the approximate solution (25) versus t (dotted curve) for $h = 0{\cdot}02$, $k = 0, 1, \ldots, 500$ (i.e., $0 \leqslant t \leqslant 10$) and $\dot{x}(0) = 0$; (a) $x_0 = x(0) = 10$, (b) $x_0 = 100$, (c) $x_0 = 500$.

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